Equations of straight lines

In this unit we find the equation of a straight line, when we are given some information about the line. The information could be the value of its gradient, together with the co-ordinates of a point on the line. Alternatively, the information might be the co-ordinates of two different points on the line. There are several different ways of expressing the final equation, and some are more general than others.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- find the equation of a straight line, given its gradient and its intercept on the y-axis;

- find the equation of a straight line, given its gradient and one point lying on it;

- find the equation of a straight line given two points lying on it;

- give the equation of a straight line in either of the forms $y = mx + c$ or $ax + by + c = 0$.
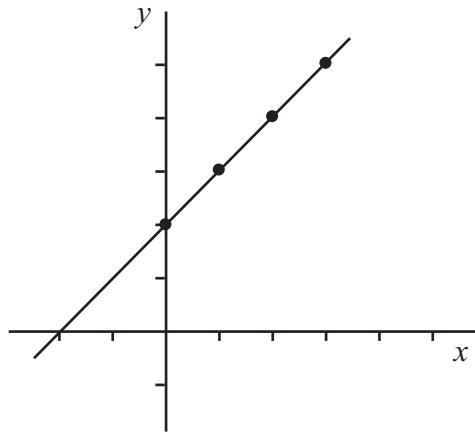
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1. Introduction

This unit is about the equations of straight lines. These equations can take various forms depending on the facts we know about the lines. So to start, suppose we have a straight line containing the points in the following list.

| x	y
0	2
1	3
2	4
3	5

There are many more points on the line, but we have enough now to see a pattern. If we take any x value and add 2, we get the corresponding y value: $0 + 2 = 2$, $1 + 2 = 3$, $2 + 2 = 4$, and so on. There is a fixed relationship between the x and y co-ordinates of any point on the line, and the equation $y = x + 2$ is always true for points on the line. We can label the line using this equation.

2. The equation of a line through the origin with a given gradient

Suppose we have a line with equation $y = x$. Then for every point on the line, the y co-ordinate must be equal to the x co-ordinate. So the line will contain points in the following list.

$y = x$:

x	y
0	0
1	1
2	2
3	3
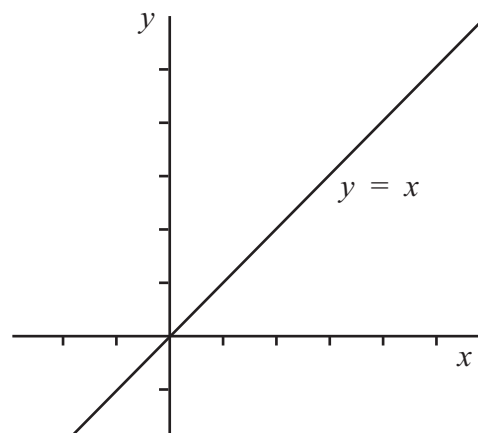
We can find the gradient of the line using the formula for gradients,

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

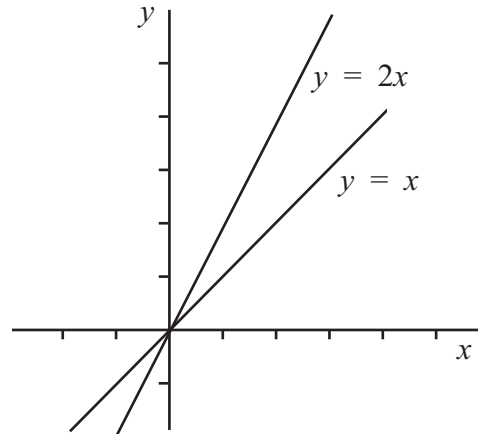and substituting in the first two sets of values from the table. We get

$$m = \frac{1 - 0}{1 - 0} = 1$$

so that the gradient of this line is 1.

What about the equation $y = 2x$? This also represents a straight line, and for all the points on the line each y value is twice the corresponding x value. So the line will contain points in the following list.

$y = 2x$:

x	y
0	0
1	2
2	4

If we calculate the gradient of the line $y = 2x$ using the first two sets of values in the table, we obtain
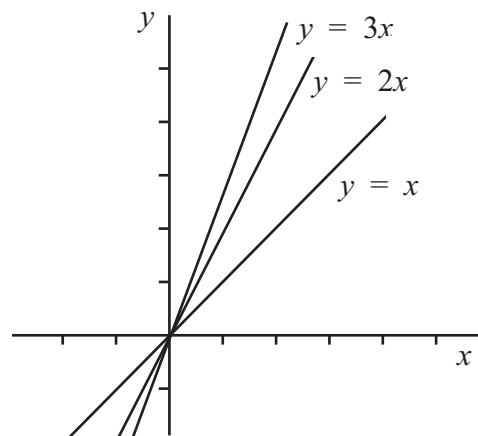
$$m = \frac{2 - 0}{1 - 0} = 2$$

so that the gradient of this line is 2.

Now take the equation $y = 3x$. This also represents a straight line, and for all the points on the line each y value is three times the corresponding x value. So the line will contain points in the following list.

$y = 3x$:

x	y
0	0
1	3
2	6

If we calculate the gradient of the line $y = 3x$ using the first two sets of values in the table, we obtain

$$m = \frac{3 - 0}{1 - 0} = 3$$

so that the gradient of this line is 3.

We can start to see a pattern here. All these lines have equations where y equals some number times x. And in each case the line passes through the origin, and the gradient of the line is given by the number multiplying x. So if we had a line with equation $y = 13x$ then we would expect the gradient of the line to be 13. Similarly, if we had a line with equation $y = -2x$ then the gradient would be -2. In general, therefore, the equation $y = mx$ represents a straight line passing through the origin with gradient m.

Key Point

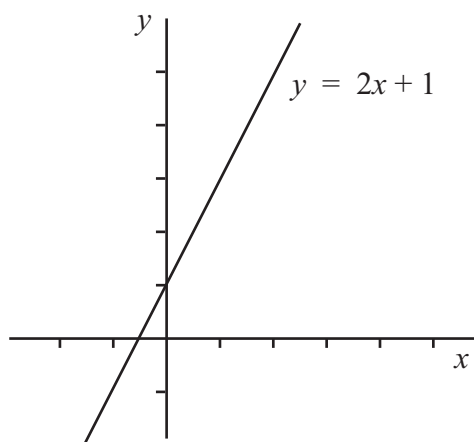The equation of a straight line with gradient m passing through the origin is given by

$$y = mx \, .$$

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3. The y-intercept of a line

Consider the straight line with equation $y = 2x + 1$. This equation is in a slightly different form from those we have seen earlier. To draw a sketch of the line, we must calculate some values.

$y = 2x + 1$:

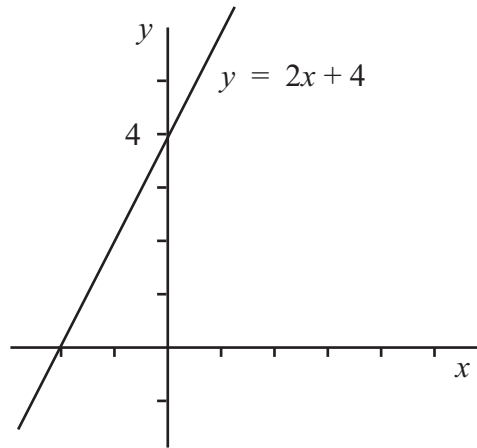| x	y
0	1
1	3
2	5

$y = 2x + 1$

Notice that when $x = 0$ the value of y is 1. So this line cuts the y-axis at $y = 1$.

What about the line $y = 2x + 4$? Again we can calculate some values.

$y = 2x + 4$:

x	y
-1	2
0	4
1	6

This line cuts the y-axis at $y = 4$.

What about the line $y = 2x - 1$? Again we can calculate some values.

$y = 2x - 1$:

x	y
-1	-3
0	-1
1	1
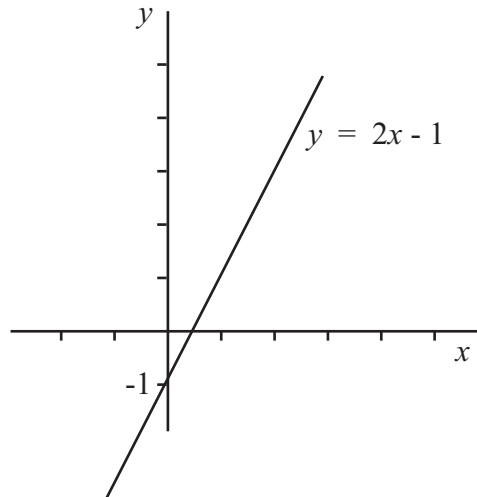
This line cuts the y-axis at $y = -1$.

The general equation of a straight line is $y = mx + c$, where m is the gradient, and $y = c$ is the value where the line cuts the y-axis. This number c is called the *intercept* on the y-axis.

Key Point

The equation of a straight line with gradient m and intercept c on the y-axis is

$$y = mx + c.$$

We are sometimes given the equation of a straight line in a different form. Suppose we have the equation $3y - 2x = 6$. How can we show that this represents a straight line, and find its gradient and its intercept value on the y-axis?

We can use algebraic rearrangement to obtain an equation in the form $y = mx + c$:

$$
\begin{aligned}
3y - 2x &= 6, \\
3y &= 2x + 6, \\
y &= \tfrac{2}{3}x + 2.
\end{aligned}
$$

So now the equation is in its standard form, and we can see that the gradient is $\frac{2}{3}$ and the intercept value on the y-axis is 2.

We can also work backwards. Suppose we know that a line has a gradient of $\frac{1}{5}$ and has a vertical intercept at $y = 1$. What would its equation be?

To find the equation we just substitute the correct values into the general formula $y = mx + c$. Here, m is $\frac{1}{5}$ and c is 1, so the equation is $y = \frac{1}{5}x + 1$. If we want to remove the fraction, we can also give the equation in the form $5y = x + 5$, or $5y - x - 5 = 0$.

Exercises

1. Determine the gradient and y-intercept for each of the straight lines in the table below.

Equation	Gradient	y-intercept
$y = 3x + 2$		
$y = 5x - 2$		
$y = -2x + 4$		
$y = 12x$		
$y = \frac{1}{2}x - \frac{2}{3}$		
$2y - 10x = 8$		
$x + y + 1 = 0$		

2. Find the equation of the lines described below (give the equation in the form $y = mx + c$):

(a) gradient 5, y-intercept 3; (b) gradient -2, y-intercept -1;
(c) gradient 3, passing through the origin; (d) gradient $\frac{1}{3}$ passing through $(0, 1)$;
(e) gradient $-\frac{3}{4}$, y-intercept $\frac{1}{2}$.

4. The equation of a straight line with given gradient, passing through a given point

Example

Suppose that we want to find the equation of a line which has a gradient of $\frac{1}{3}$ and passes through the point $(1, 2)$. Here, whilst we know the gradient, we do not know the value of the y-intercept c.

We start with the general equation of a straight line $y = mx + c$.

We know the gradient is $\frac{1}{3}$ and so we can substitute this value for m straightaway. This gives $y = \frac{1}{3}x + c$.

We now use the fact that the line passes through $(1, 2)$. This means that when $x = 1$, y must be 2. Substituting these values we find

$$2 = \frac{1}{3}(1) + c$$

so that

$$c = 2 - \frac{1}{3} = \frac{5}{3}$$

So the equation of the line is $y = \frac{1}{3}x + \frac{5}{3}$.

We can work out a general formula for problems of this type by using the same method. We shall take a general line with gradient m, passing through the fixed point $A(x_1, y_1)$.

We start with the general equation of a straight line $y = mx + c$.

We now use the fact that the line passes through $A(x_1, y_1)$. This means that when $x = x_1$, y must be y_1. Substituting these values we find

$$y_1 = mx_1 + c$$

so that

$$c = y_1 - mx_1$$

So the equation of the line is $y = mx + y_1 - mx_1$.

We can write this in the alternative form

$$y - y_1 = m(x - x_1)$$

This then represents a straight line with gradient m, passing through the point (x_1, y_1). So this general form is useful if you know the gradient and one point on the line.

 Key Point

The equation of a straight line with gradient m, passing through the point (x_1, y_1), is

$$y - y_1 = m(x - x_1).$$

For example, suppose we know that a line has gradient -2 and passes through the point $(-3, 2)$. We can use the formula $y - y_1 = m(x - x_1)$ and substitute in the values straight away:

$$\begin{aligned} y - 2 &= -2(x - (-3)) \\ &= -2(x + 3) \\ &= -2x - 6 \\ y &= -2x - 4 \,. \end{aligned}$$

Exercise

3. Find the equation of the lines described below (give the equation in the form $y = mx + c$):

(a) gradient 3, passing through $(1, 4)$; (b) gradient -2, passing through $(2, 0)$;
(c) gradient $\frac{2}{5}$, passing through $(5, -1)$; (d) gradient 0, passing $(-1, 2)$;
(e) gradient -1, passing through $(1, -1)$.

5. The equation of a straight line through two given points

What should we do if we want to find the equation of a straight line which passes through the two points $(-1, 2)$ and $(2, 4)$?

Here we don't know the gradient of the line, so it seems as though we cannot use any of the formulæ we have found so far. But we do know two points on the line, and so we can use them to work out the gradient. We just use the formula $m = (y_2 - y_1)/(x_2 - x_1)$. We get

$$m = \frac{4 - 2}{2 - (-1)} = \frac{2}{3} \,.$$

So the gradient of the line is $\frac{2}{3}$. And we know two points on the line, so we can use one of them in the formula $y - y_1 = m(x - x_1)$. If we take the point $(2, 4)$ we get

$$\begin{aligned} y - 4 &= \tfrac{2}{3}(x - 2) \\ 3y - 12 &= 2x - 4 \\ 3y &= 2x + 8 \\ y &= \tfrac{2}{3}x + \tfrac{8}{3} \,. \end{aligned}$$

As before, it will be useful to find a general formula that can be used for examples of this kind. So suppose the general line passes through two points $A(x_1, y_1)$ and $B(x_2, y_2)$. We shall let a general point on the line be $P(x, y)$.
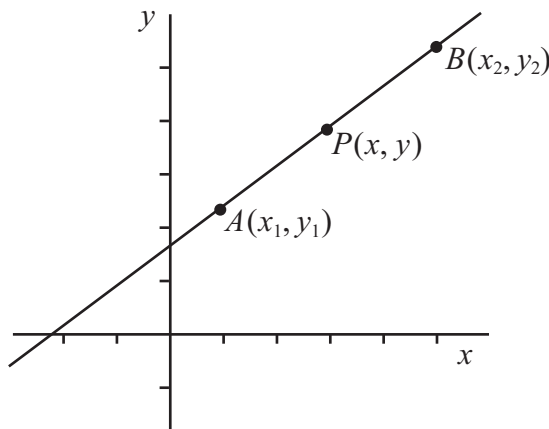
Now we know that the gradient of AP must be the same as the gradient of AB, as all three points are on the same line. But the gradient of AP is

$$m_{AP} = \frac{y - y_1}{x - x_1},$$

whereas the gradient of AB is

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Then $m_{AP} = m_{AB}$, so we must have

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Now this formula is fairly complicated, but it is easier to remember if all the terms involving y are on one side, and all the terms involving x are on the other. If we manipulate the formula, we get first

$$y - y_1 = (x - x_1)\frac{y_2 - y_1}{x_2 - x_1}$$

and then

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

It might help you to remember this formula if you notice that the pattern on the left-hand side, involving y, is just the same as the pattern on the right-hand side, involving x.

Key Point

The equation of a straight line passing through the two points (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

Now we can use this formula for an example. Suppose that we want to find the equation of the straight line which passes through the two points $(1, -2)$ and $(-3, 0)$. We just substitute into the formula, and rearrange. The various steps are

$$\begin{aligned}
\frac{y - (-2)}{0 - (-2)} &= \frac{x - 1}{-3 - 1} \\
\frac{y + 2}{2} &= \frac{x - 1}{-4} \\
y + 2 &= \frac{x - 1}{-2} \\
&= -\tfrac{1}{2}(x - 1) \\
-2y - 4 &= x - 1 \\
-2y &= x + 3 \\
y &= -\tfrac{1}{2}x - \tfrac{3}{2}.
\end{aligned}$$

So the line has gradient $-\frac{1}{2}$ and its intercept on the y-axis is $-\frac{3}{2}$. We can also rearrange the equation a little further to obtain $2y = -x - 3$, or $2y + x + 3 = 0$.

Exercise

4. Find the equation of the lines described below (give the equation in the form $y = mx + c$):

(a) passing through $(4, 6)$ and $(8, 26)$, (b) passing through $(1, 1)$ and $(4, -8)$,
(c) passing through $(3, 4)$ and $(5, 4)$,   (d) passing through $(0, 2)$ and $(4, 0)$,
(e) passing through $(-2, 3)$ and $(2, -5)$.

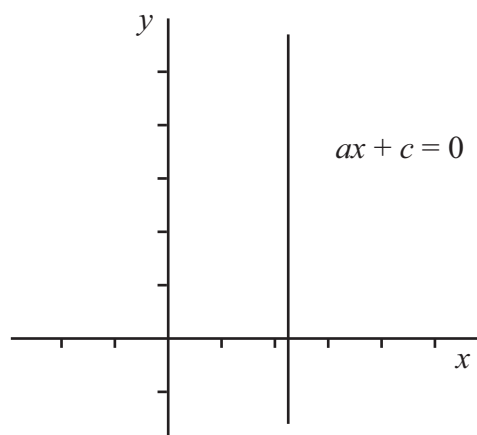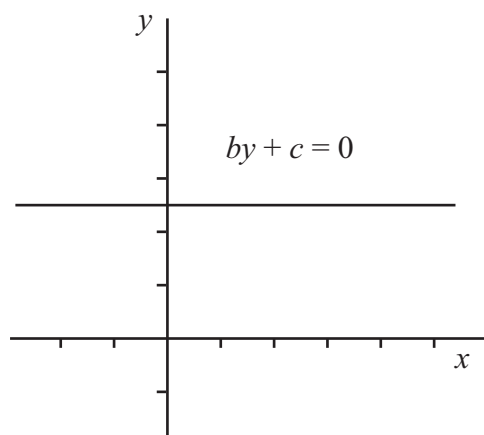# 6. The most general equation of a straight line

There is one more form of the equation for a straight line that is sometimes needed. This is the equation

$$ax + by + c = 0 \,.$$

We have written equations in this form for some of our examples. We can see some special cases of this equation by setting either a or b equal to zero.

If $a = 0$ then we obtain lines with general equation $by + c = 0$, i.e. $y = -\frac{c}{b}$. These lines are horizontal, so that they are parallel to the x-axis.

If $b = 0$ then we obtain lines with general equation $ax + c = 0$, i.e. $x = -\frac{c}{a}$. These lines are vertical, so that they are parallel to the y-axis. The equation of a vertical line cannot be written in the form $y = mx + c$. The equation $ax + by + c = 0$ is the most general equation for a straight line, and can be used where other forms of equation are not suitable.

Key Point

The most general equation of a straight line is

$$ax + by + c = 0 \,.$$

If $a = 0$ then the line is horizontal, and if $b = 0$ then the line is vertical.

Exercise 5. Find the equation of the lines described below (give the equation in the form $ax + by + c = 0$, where a, b and c are whole numbers and $a > 0$):

(a)   the line in Exercise 2 (b)
(b)   the line in Exercise 2 (e)
(c)   the line in Exercise 3 (c)
(d)   the line in Exercise 4 (b)
(e)   the line in Exercise 4 (d)
(f)   the line in Exercise 4 (e)
(g)   the line through $(3, -2)$ and $(3, 2)$
(h)   the vertical line passing through the point $\left(0, \frac{2}{3}\right)$.

Answers

1.

Equation	Gradient	y-intercept
$y = 3x + 2$	3	2
$y = 5x - 2$	5	-2
$y = -2x + 4$	-2	4
$y = 12x$	12	0
$y = \frac{1}{2}x - \frac{2}{3}$	$\frac{1}{2}$	$-\frac{2}{3}$
$2y - 10x = 8$	5	4
$x + y + 1 = 0$	-1	-1

2.

(a) $y = 5x + 3$,   (b) $y = -2x - 1$,   (c) $y = 3x$,   (d) $y = \frac{1}{3}x + 1$,   (e) $y = -\frac{3}{4}x + \frac{1}{2}$.

3.

(a) $y = 3x + 1$,   (b) $y = -2x + 4$,   (c) $y = \frac{2}{5}x - 3$,   (d) $y = 2$,   (e) $y = -x$.

4.

(a) $y = 5x - 14$,   (b) $y = -3x + 4$,   (c) $y = 4$,   (d) $y = -\frac{1}{2}x + 2$,   (e) $y = -2x - 1$.

5.

(a) $2x + y + 1 = 0$,   (b) $3x + 4y - 2 = 0$,   (c) $2x - 5y - 15 = 0$,   (d) $3x + y - 4 = 0$,
(e) $x + 2y - 4 = 0$,   (f) $2x + y + 1 = 0$,   (g) $x - 3 = 0$,   (h) $x = 0$.